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# A note on a breakdown of the multiplicative composition of inner and outer expansions

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It is shown that the multiplicative rule of the method of matched asymptotic expansions fails under certain conditions. The velocity distribution in front of an elliptic airfoil is given as an example. The reason for the breakdown is explained by inspecting the usual formal justification of multiplicative composition.

## 1. Introduction

When a problem has been attacked by the method of matched asymptotic expansions and the outer solution as well as the inner solution has already been found, one then wishes to find a single uniformly valid solution. This is usually done by constructing a composite expansion by means of either additive or multiplicative composition. If outer and inner expansions are denoted by subscripts $o$ and $i$, respectively, while the number of terms is denoted by superscripts $n$ or $m$, the rule for additive composition can be formally written as

$$f_{+}^{(m,n)} = f_i^{(m)} + f_o^{(n)} - [f_o^{(n)}]_i^{(m)} \tag{1}$$

whereas multiplicative composition is performed by

$$f_{\times}^{(m,n)} = f_i^{(m)} f_o^{(n)} / [f_o^{(n)}]_i^{(m)}, \tag{2}$$

cf. Van Dyke (1964, pp. 94–97). Both methods have been in use now for years. Nevertheless, the following simple example will show that, under certain circumstances, the multiplicative rule yields incorrect results.

## 2. Velocity distribution in front of an elliptic airfoil

In his well-known book, Van Dyke (1964, pp. 62–68) introduces the method of matched asymptotic expansions by examining the velocity $q$ on the surface of a thin elliptic airfoil of thickness ratio $\epsilon$ in steady incompressible flow. This example can readily be extended to include the velocity on the axis of symmetry in front of (and behind) the airfoil. In agreement with Van Dyke's notation let $s$ be the distance measured from the leading edge in the direction of the free-stream velocity $U$ and normalized such that the length of the elliptic airfoil is 2. Then, for $s < 0$, the result of the two-term outer expansion (formal thin-airfoil series) is

$$(q/U)_o^{(2)} = 1 + \epsilon\{1 - [1 + (s^2 - 2s)^{-1}]^{\frac{1}{2}}\}. \tag{3}$$

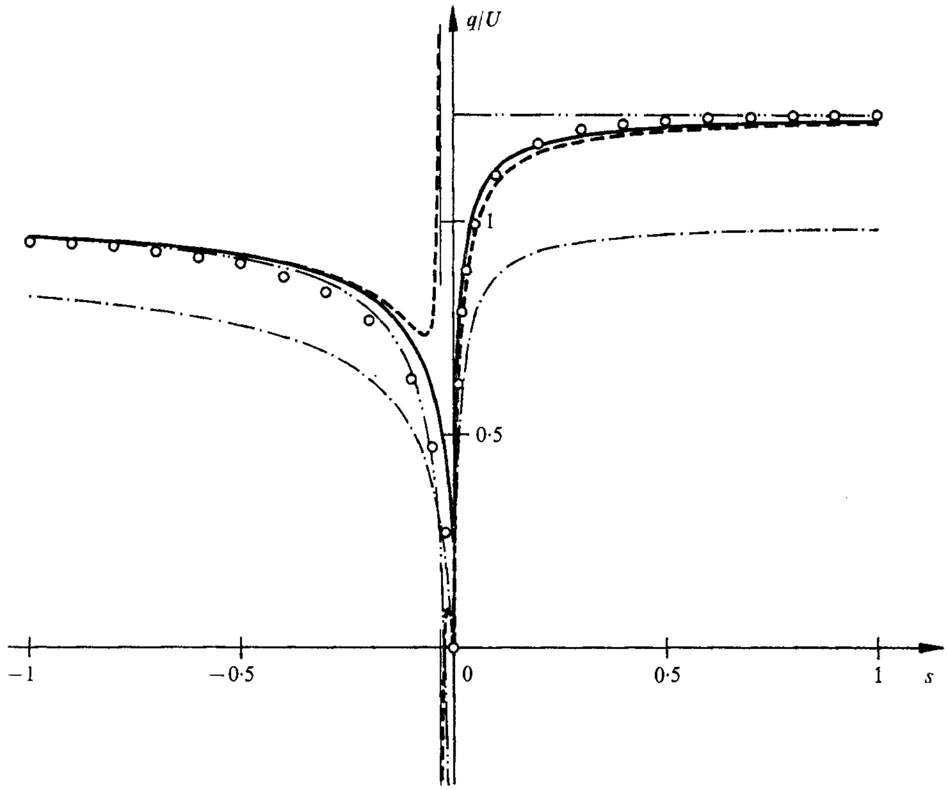


FIGURE 1. Velocity distribution on the stagnation streamline of a thin elliptic airfoil in steady incompressible flow. Comparison of the one-term inner, the two-term outer and the corresponding composite expansions with the exact solution. (Thickness ratio $\epsilon = 0{\cdot}25$.) ○, exact solution; —··—, two-term outer expansion; —·—, one-term inner expansion; ——, corresponding additive composite expansion; ---, corresponding multiplicative composite expansion.

The one-term inner expansion (local solution near the leading edge) for $S < 0$ becomes

$$(q/U)_i^{(1)} = 1-(1-2S)^{-\frac{1}{2}}, \tag{4}$$

where the inner co-ordinate $S$ is related to the outer co-ordinate $s$ according to

$$S = s/\epsilon^2. \tag{5}$$

Additive composition of the expansions (3) and (4) yields

$$(q/U)_+^{(1,2)} = 1+\epsilon\{1-[1+(s^2-2s)^{-1}]^{\frac{1}{2}}+(-2s)^{-\frac{1}{2}}-(\epsilon^2-2s)^{-\frac{1}{2}}\} \tag{6}$$

and the result of multiplicative composition is

$$(q/U)_\times^{(1,2)} = \{1+\epsilon-\epsilon[1+(s^2-2s)^{-1}]^{\frac{1}{2}}\}[1-\epsilon(\epsilon^2-2s)^{-\frac{1}{2}}][1-\epsilon(-2s)^{-\frac{1}{2}}]^{-1}. \tag{7}$$

Numerical results for $\epsilon = 0{\cdot}25$ have been plotted in figure 1, which gives an overall picture, and in figure 2 which shows the local behaviour of the solutions near the stagnation point. For convenience the velocity distribution at the

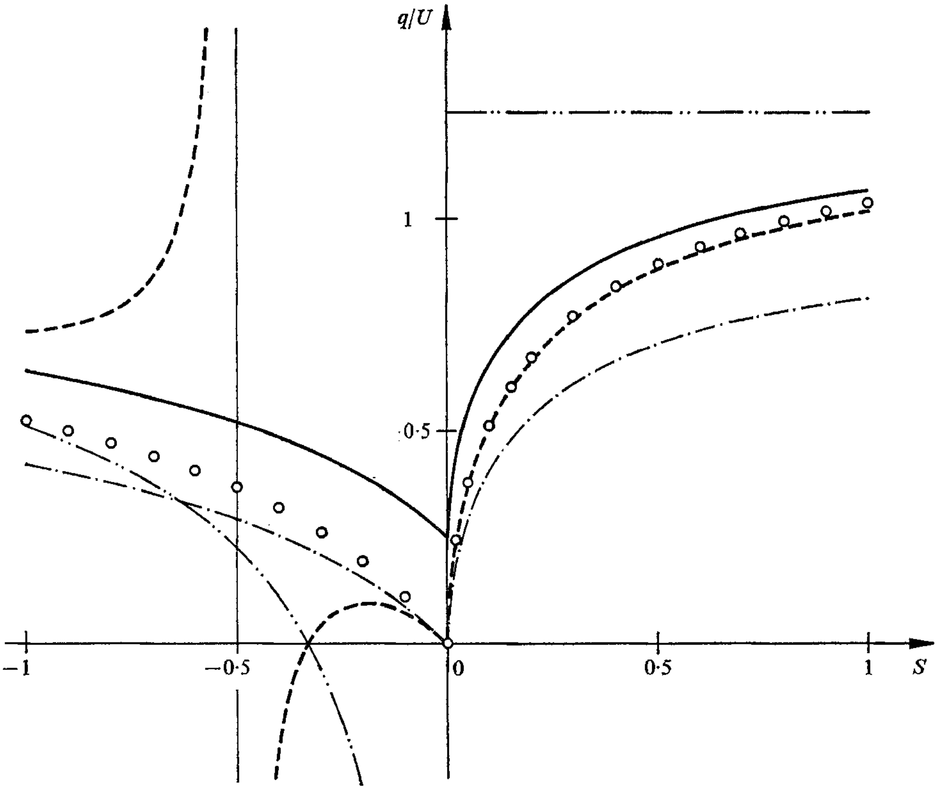


FIGURE 2. Velocity distribution on the stagnation streamline near the leading edge of the elliptic airfoil. Comparison of the one-term inner, the two-term outer, and the corresponding composite expansions with the exact solution. (Thickness ratio $\epsilon = 0{\cdot}25$.) Notation as in figure 1.

surface ($s > 0, S > 0$) is also given in the figures though it shows no peculiarities. Comparison is made with the exact solution

$$(q/U)_{\text{ex}} = (1-\epsilon)^{-1}\{1+\epsilon(s-1)\,[(s-1)^2-1+\epsilon^2]^{-\frac{1}{2}}\} \quad (\text{for } s < 0), \tag{8}$$

which can easily be extracted from known results (cf. Lamb 1932, pp. 84–85).

Figures 1 and 2 show that, since an error of order $\epsilon$ has to be tolerated at the stagnation point in this approximation, the additive solution (6) is a useful, uniformly valid approximation. The multiplicative solution (7), however, has a singularity at the point

$$s = -\tfrac{1}{2}\epsilon^2. \tag{9}$$

Obviously the multiplicative rule does not lead to a uniformly valid solution in our example. This is the more disappointing as *on* the airfoil surface the multiplicative solution happens to be more accurate than the additive solution.

We might try to improve the composite expansions by including higher order terms. The two-term inner expansion for $S < 0$ is

$$(q/U)_i^{(2)} = (1+\epsilon)\,[1-(1-2S)^{-\frac{1}{2}}]. \tag{10}$$

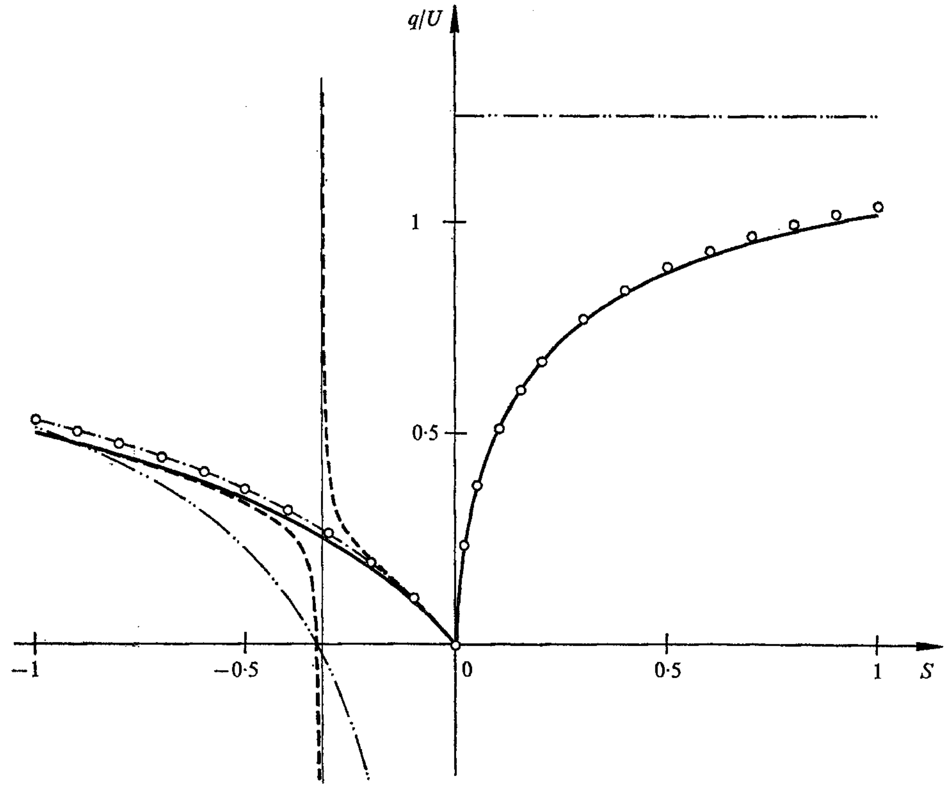


FIGURE 3. Velocity distribution on the stagnation streamline near the leading edge of the elliptic airfoil. Comparison of the two-term inner, the two-term outer and the corresponding composite expansions with the exact solution. (Thickness ratio $\epsilon = 0{\cdot}25$.) ○, exact solution; —··—, two-term outer expansion; —·—, two-term inner expansion; ——, corresponding additive composite expansion; – – –, corresponding multiplicative composite expansion.

Hence the corresponding composite expansions become

$$(q/U)_{+}^{(2,2)} = 1+\epsilon\{1-[1+(s^2-2s)^{-1}]^{\frac{1}{2}}+(-2s)^{-\frac{1}{2}}-(1+\epsilon)(\epsilon^2-2s)^{-\frac{1}{2}}\}; \tag{11}$$

$$(q/U)_{\times}^{(2,2)} = \{1+\epsilon-\epsilon[1+(s^2-2s)^{-1}]^{\frac{1}{2}}\} \times (1+\epsilon)[1-\epsilon(\epsilon^2-2s)^{-\frac{1}{2}}][1+\epsilon-\epsilon(-2s)^{-\frac{1}{2}}]^{-1}. \tag{12}$$

Numerical results are given in figure 3. The multiplicative solution again has a singularity, which is now at the point

$$s = -\tfrac{1}{2}[\epsilon/(1+\epsilon)]^2. \tag{13}$$

Note that in both cases the singularity is located within the inner region, i.e. $S = O(1)$.

## 3. Failure of the formal justification of the multiplicative rule

The composition rules (1) and (2) are usually justified by formally showing that by taking the inner (or outer) expansion of the composite solution the inner (or outer) solution is recovered. Specifically, carrying out the inner expansion on both sides of (2) yields

$$[f_{\times}^{(m,n)}]_i^{(m)} = f_i^{(m)}\{[f_0^{(n)}]_i^{(m)} + \text{higher order terms}\}/[f_o^{(n)}]_i^{(m)}. \tag{14}$$

Now, if the so-called common part of the inner and outer expansions, i.e. $[f_o^{(n)}]_i^{(m)}$, is non-zero for all possible values of the independent variables, the higher order terms in (14) can be disregarded. Hence the common part cancels, and the inner solution is obtained as it should be in order to make multiplicative composition valid in the inner region. If, however, $[f_o^{(n)}]_i^{(m)}$ vanishes for certain values of the independent variables, the higher order terms in (14) have to be retained. In this case the desired result, i.e. $f_i^{(m)}$, is not obtained from (14). Moreover, the inner expansion of the multiplicative solution generally goes to infinity as the common part goes to zero since it is unlikely that the higher order terms in (14) will also vanish for the same values of the independent variables. This explains the singularities we encountered in the example of the elliptic airfoil.

## 4. Conclusions

Multiplicative composition of inner and outer expansions generally fails to provide a uniformly valid solution if the common part of the inner and outer expansions has zeros. Use of additive composition is recommended to avoid the danger of such a breakdown.

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